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ALMOST PERIODICITY IN DIFFERENTIAL EQUATIONS

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Abstract. We study the existence of S-asymptotically ω-periodic mild solutions of partial functional integrodifferential equation by the method of Caicedo et al. [2].

1. Introduction

The study of the existence of almost periodic, S-asymptotically ω-periodic, asymptotically almost periodic, pseudo almost periodic and almost automorphic solutions is one of the most important topics in the qualitative theory of differential equations both due to its mathematical interest as well as due to their applications in physics and mathematical biology, among areas.

We study the existence of S-asymptotically ω-periodic mild solutions of the following partial functional integro-differential equation

$$\begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t,u_t), \ t \geq 0, \\ u_0 = \varphi \in C = C([-r,0],X), \end{cases} \tag{1.1}$$

where $A : D(A) \subseteq X \to X$ is the infinitesimal generator of a C_0-semigroup on a Banach space X, for $t \geq 0$, $B(t)$ is a closed linear operator with domain $D(B(t)) \supset D(A)$, C denotes the space of continuous functions from $[-r,0]$ to X endowed with the uniform norm topology, for every $t \geq 0$, u_t denotes the history function of C defined by

$$u_t(\theta) = u(t+\theta), -r \leq \theta \leq 0,$$

$f : \mathbb{R}_+ \times C \to X$ is a continuous function.

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In [5], the authors established the global existence and blowing up of the mild solution $u : [-r, +\infty) \to X$ which is continuous and satisfies

$$\begin{cases} u(t) = R(t)\varphi(0) + \int_0^t R(t-s) f(s, u_s) ds, \ t \geq 0, \\ u_0 = \varphi, \end{cases} \tag{1.2}$$

where $R(t)$ is the resolvent operator for the linear homogeneous equation

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s) v(s) ds, \ t \geq 0, \\ v(0) = v_0 \in X. \end{cases} \tag{1.3}$$

Caicedo et al. [2] studied the existence of S-asymptotically ω-periodic solutions for a class of the equation

$$\begin{cases} u'(t) = Au(t) + \int_0^t B(t-s) u(s) ds + g(t, u(t)), \ t \geq 0, \\ u_0 = x_0 \in X, \end{cases} \tag{1.4}$$

where $g : \mathbb{R}_+ \times X \to X$ is a suitable function.

Caicedo and Cuevas [1] investigated the existence and uniqueness of S-asymptotically ω-periodic mild solutions to the abstract partial integrodifferential equation with infinite delay

$$\begin{cases} \frac{d}{dt} D(t, u_t) = AD(t, u_t) + \int_0^t B(t-s) D(s, u_s) ds + g(t, u_t), t \geq 0, \\ u_0 = \varphi \in \mathcal{B}, \end{cases} \tag{1.5}$$

where $D(t, \varphi) = \varphi(0) + f(t, \varphi)$ and u_t belongs to an abstract phase space $\mathcal{B}$ by Hale and Kato [9].

Also, Dimbour and N'Guerekata [4] obtained some results about the existence of S-asymptotically ω-periodic mild solutions of the equation

$$\begin{cases} \frac{d}{dt}[u(t) + f(t, Bu(t))] = Au(t) + g(t, Cu(t)), \ t \in \mathbb{R}, \\ u(0) = 0, \end{cases} \tag{1.6}$$

where B, C are two densely defined closed linear operators on X and f, g are continuous functions.

2. S-asymptotically ω-periodic mild solutions

It is convenient to introduce notion: For a Banach space X,

$$C(\mathbb{R}_+, X) = \{x : \mathbb{R}_+ \to X \ : \ x \text{ is continuous}\},$$
$$C_b(\mathbb{R}_+, X) = \{x \in C(\mathbb{R}_+, X) \ : \ \sup_{t \geq 0} ||x(t)|| < \infty\},$$
$$C_0(\mathbb{R}_+, X) = \{x \in C_b(\mathbb{R}_+, X) \ : \ \lim_{t \to \infty} ||x(t)|| = 0\},$$
$$C_\omega(\mathbb{R}_+, X) = \{x \in C_b(\mathbb{R}_+, X) \ : \ x \text{ is } \omega-\text{periodic}\}.$$

We consider Eq. (1.1).

$$\begin{cases} u'(t) = Au(t) + \int_0^t B(t-s)u(s)ds + f(t, u_t), \ t \geq 0, \\ u_0 = \varphi \in C, \end{cases}$$

and Eq. (1.3)

$$\begin{cases} v'(t) = Av(t) + \int_0^t B(t-s)v(s)ds, \ t \geq 0, \\ v(0) = v_0 \in X. \end{cases}$$

In the following Y denotes the Banach space $D(A)$ equipped with the graph norm defined by

$$||y||_Y = ||Ay|| + ||y||, y \in Y.$$

Definition 2.1 ([6]). A *resolvent operator* for Eq. (1.3) is a bounded linear operator valued function $R(t)$ for $t \geq 0$ such that

(i) $R(0) = I$(identity operator) and $||R(t)|| \leq Me^{-\mu t}$ for some constants $M > 0$ and $\mu > 0$.
(ii) For all $x \in X$, $R(t)x$ is strongly continuous for $t \geq 0$.
(iii) For $x \in Y$, $R(\cdot)x \in C^1(\mathbb{R}_+, X) \cap C(\mathbb{R}_+, Y)$ and

$$\begin{aligned} R'(t)x &= AR(t)x + \int_0^t B(t-s)R(s)xds \\ &= R(t)Ax + \int_0^t R(t-s)B(s)xds, t \geq 0. \end{aligned}$$

We assume that there exists a unique resolvent operator for Eq. (1.3).

Definition 2.2 ([5]). We say that a continuous function $u : [-r, +\infty) \to X$ is a *mild solution* of Eq. (1.1) if u satisfies

$$\begin{cases} u(t) = R(t)\varphi(0) + \int_0^t R(t-s)f(s, u_s)ds, t \geq 0, \\ u_0 = \varphi \in C([-r, 0], X). \end{cases}$$

Ezzinbi et al. [5] established the following global existence and blowing up of the mild solutions for Eq. (1.1).

Theorem 2.3. *Assume that f is locally Lipschitz, i.e.,*

$$||f(t,\varphi_1) - f(t,\varphi_2)|| \le c_0(\alpha)||\varphi_1 - \varphi_2||$$

for some constant $c_0(\alpha)$, for all $\varphi_1, \varphi_2 \in C$ with $||\varphi_1||, ||\varphi_2|| \le \alpha, \alpha > 0$. Then there exist a maximal interval of existence $[-r, b_\varphi)$ and a unique mild solution of Eq. (1.1) defined on $[-r, b_\varphi)$ and either

$$b_\varphi = +\infty \text{ or } \limsup_{t \to b_\varphi^-} ||u(t,\varphi)|| = +\infty.$$

Moreover, $u(t,\varphi)$ is a continuous function of φ in the sense that if $\varphi \in C$ and $t \in [0, b_\varphi)$, then there exist positive constants k and ε such that, for $\psi \in C$ and $||\varphi - \psi|| < \varepsilon$, we have

$$t \in [0, b_\psi) \text{ and } ||u(s,\varphi) - u(s,\psi)|| \le k||\varphi - \psi||, s \in [-r, t].$$

We next recall some concepts concerning almost periodic functions in [8].

Definition 2.4. A function $f \in C(\mathbb{R}, X)$ is said to be *almost periodic* if for each $\varepsilon > 0$, there exists an $l(\varepsilon) > 0$ such that every interval of length $l(\varepsilon)$ contains a number τ with the property that

$$||f(t+\tau) - f(t)|| < \epsilon, \; t \in \mathbb{R}.$$

Denote by $AP(\mathbb{R}, X)$ the set of such functions.

Definition 2.5. A function $f \in C(\mathbb{R}_+, X)$ is called *S-asymptotically ω-periodic* if there exists an $\omega > 0$ such that

$$\lim_{t \to \infty} [f(t+\omega) - f(t)] = 0.$$

In this case, we say that ω is an asymptotic period of f and that f is S-asymptotically ω-periodic. Denote by $SAP_\omega(X)$ the set of such functions.

Definition 2.6. A function $f \in C_b(\mathbb{R}_+, X)$ is called *asymptotically almost periodic* if there exist $g \in AP(\mathbb{R}, X)$ and $\varphi \in C_0(\mathbb{R}_+, X)$ such that $f = g + \varphi$. Also, f is said to be *asymptotically ω-periodic* when $g \in C_\omega(\mathbb{R}_+, X)$.

Example 2.7. [10] *A function $f : \mathbb{R}_+ \to \mathbb{R}$ defined by*

$$f(t) = \sin(\ln(t+1)), t \in \mathbb{R}_+,$$

is S-asymptotically ω-periodic for any $\omega > 0$, but not asymptotically ω-periodic.

In the following, W and Z are Banach spaces.

Definition 2.8. A function $F \in C(\mathbb{R}_+ \times Z, W)$ is called *uniformly S-asymptotically ω-periodic* on bounded sets if $F(\cdot, x)$ is bounded for each $x \in Z$, and for every $\varepsilon > 0$ and all bounded sets $K \subseteq Z$ there exists $L_{K,\varepsilon} \geq 0$ such that

$$||F(t+\omega, x) - F(t,x)||_W \leq \varepsilon,\ t \geq L_{K,\varepsilon}, x \in K.$$

Definition 2.9. A function $F \in C(\mathbb{R}_+ \times Z, W)$ is called *asymptotically uniformly continuous* on bounded sets if for every $\varepsilon > 0$ and all bounded sets $K \subseteq Z$ there exist constants $L_{K,\varepsilon} \geq 0$ and $\delta_{K,\varepsilon} > 0$ such that

$$||F(t,x) - F(t,y)||_W \leq \varepsilon,\ t \geq L_{K,\varepsilon},$$

when $||x-y||_Z \leq \delta_{K,\varepsilon},\ x, y \in K$.

Lemma 2.10. [7] *Assume that $F : \mathbb{R}_+ \times Z \to W$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Then*

$$\lim_{t\to\infty} [F(t+\omega, u(t+\omega)) - F(t, u(t))] = 0.$$

Lemma 2.11. [7] *Let $u : \mathbb{R} \to X$ be a continuous function with $u_0 \in C$ and $u|_{\mathbb{R}_+} \in SAP_\omega(X)$. Then the function $t \to u_t$ belongs to $SAP_\omega(X)$.*

The following result was obtained by Caicedo et al. [2]. For the completeness we give a proof in detail.

Theorem 2.12. *For Eq. (1.4), assume that $g : \mathbb{R}_+ \times X \to X$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Let $u \in SAP_\omega(X)$. If there exists a constant $L > 0$ such that*

$$||g(t,x) - g(t,y)|| \leq L||x-y||,\ t \geq 0, x, y \in X$$

and $\frac{LM}{\mu} < 1$, then there exists a unique S-asymptotically ω-periodic mild solution of Eq. (1.4).

Proof. Define the operator $\Gamma : SAP_\omega(X) \to SAP_\omega(X)$ by

$$\Gamma u(t) = R(t)x_0 + \int_0^t R(t-s) g(s, u(s)) ds,\ t \geq 0.$$

By Lemma 2.10, $g(\cdot, u(\cdot))$ belongs to $SAP_\omega(X)$.

Let

$$v(t) = \int_0^t R(t-s) g(s, u(s)) ds.$$

Then

$$\begin{aligned}||v(t)|| &\leq \int_0^t ||R(t-s)||||g(s,u(s))||ds \\ &\leq \int_0^t Me^{-\mu(t-s)}||g(s,u(s))||ds \\ &\leq \frac{M}{\mu}\int_0^t ||g(s,u(s))||ds.\end{aligned}$$

Thus

$$||v||_\infty = \sup_{t\geq 0}||v(t)|| \leq \frac{M}{\mu}||g||_\infty.$$

Hence $v \in C_b(\mathbb{R}_+, X)$.

Note that for any $\varepsilon > 0$, there exists $T > 0$ such that

$$||u(t+\omega) - u(t)|| \leq \varepsilon,\ t \geq T,$$

$$\int_T^\infty e^{-\mu s}ds \leq \varepsilon.$$

Then

$$\begin{aligned}&v(t+\omega) - v(t) \\ &= \int_0^{t+\omega} R(t+\omega-s)g(s,u(s))ds - \int_0^t R(t-s)g(s,u(s))ds \\ &= \int_0^{\omega} R(t+\omega-s)g(s,u(s))ds + \int_\omega^{t+\omega} R(t+\omega-s)g(s,u(s))ds \\ &\quad - \int_0^t R(t-s)g(s,u(s))ds \\ &= \int_t^{t+\omega} R(s)g(t+\omega-s,u(t+\omega-s))ds \\ &\quad + \int_0^t R(t-s)[g(s+\omega,u(s+\omega)) - g(s,u(s))]ds \\ &= \int_t^{t+\omega} R(s)g(t+\omega-s,u(t+\omega-s))ds \\ &\quad + \int_0^T R(t-s)[g(s+\omega,u(s+\omega)) - g(s,u(s))]ds \\ &\quad + \int_T^t R(t-s)[g(s+\omega,u(s+\omega)) - g(s,u(s))]ds.\end{aligned}$$

For any $t \geq 2T$, we have

$$
\begin{aligned}
&||v(t+\omega)-v(t)|| \\
&\leq M||g||_\infty \int_t^{t+\omega} e^{-\mu s} ds + 2M||g||_\infty \int_{t-T}^{t} e^{-\mu s} ds + \varepsilon LM||u||_\infty \int_0^{t-T} e^{-\mu s} ds \\
&\leq M||g||_\infty \int_t^{t+\omega} e^{-\mu s} ds + 2M||g||_\infty \int_{T}^{t} e^{-\mu s} ds + \varepsilon LM||u||_\infty \int_0^{t} e^{-\mu s} ds \\
&\leq 3M||g||_\infty \int_T^{\infty} e^{-\mu s} ds + \frac{LM\varepsilon}{\mu} \\
&= M(3||g||_\infty + \frac{L}{\mu}||u||_\infty)\varepsilon.
\end{aligned}
$$

Therefore $\Gamma \in SAP_\omega(X)$ since $R(\cdot)x_0 \in SAP_\omega(X)$.

Now, we show that Γ is a contraction. For $u_1, u_2 \in SAP_\omega(X)$,

$$
\begin{aligned}
||\Gamma u_1(t) - \Gamma u_2(t)|| &\leq \int_0^t ||R(t-s)[g(s,u_1(s)) - g(s,u_2(s))]||ds \\
&\leq LM \int_0^t e^{-\mu(t-s)}||u_1(s) - u_2(s)||ds \\
&\leq \frac{LM}{\mu}||u_1 - u_2||_\infty.
\end{aligned}
$$

Since $\frac{LM}{\mu} < 1$, Γ is a contraction. Hence there exists a unique fixed point u of Γ and u is the S-asymptotically ω-periodic mild solution of Eq. (1.4). □

Now, we investigate the existence of S-asymptotically ω-periodic mild solutions of Eq. (1.1).

LEMMA 2.13. *Assume that $f : \mathbb{R}_+ \times C \to X$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. If $u \in SAP_\omega(X)$, then*

$$
v(t) = \int_0^t R(t-s) f(s, u_s) ds
$$

belongs to $SAP_\omega(X)$.

Proof. We have

$$
\begin{aligned}
||v||_\infty &\leq \sup_{t\geq 0} \int_0^t M e^{-\mu(t-s)}||f(s,u_s)||ds \\
&\leq \frac{M}{\mu}||f||_\infty < +\infty.
\end{aligned}
$$

It follows that $v \in C_b(\mathbb{R}_+, X)$.

Note that the function $t \to u_t$ and $f(s, u_s)$ belong to $SAP_\omega(X)$ by Lemmas 2.10 and 2.11. For every $\varepsilon > 0$, there exists $T > 0$ such that

$$||u(t+\omega) - u(t)|| \le \varepsilon, \; t \ge T,$$
$$\int_T^\infty e^{-\mu s} ds \le \varepsilon.$$

Then

$$\begin{aligned}
& v(t+\omega) - v(t) \\
&= \int_0^{t+\omega} R(t+\omega-s) f(s, u_s)) ds - \int_0^t R(t-s) f(s, u_s) ds \\
&= \int_0^\omega R(t+\omega-s) f(s, u_s) ds + \int_\omega^{t+\omega} R(t+\omega-s) f(s, u_s) ds \\
&\quad - \int_0^t R(t-s) f(s, u_s) ds \\
&= \int_t^{t+\omega} R(s) f(t+\omega-s, u_{t+\omega-s}) ds \\
&\quad + \int_0^t R(t-s)[f(s+\omega, u_{s+\omega}) - f(s, u_s)] ds \\
&= \int_t^{t+\omega} R(s) f(t+\omega-s, u_{t+\omega-s}) ds \\
&\quad + \int_0^T R(t-s)[f(s+\omega, u_{s+\omega}) - f(s, u_s)] ds \\
&\quad + \int_T^t R(t-s)[f(s+\omega, u_{s+\omega}) - f(s, u_s)] ds.
\end{aligned}$$

Hence, by the same calculation of the proof in Theorem 2.12, we get the result. □

THEOREM 2.14. *Assume that $f : \mathbb{R}_+ \times C \to X$ is uniformly S-asymptotically ω-periodic on bounded sets and asymptotically uniformly continuous on bounded sets. Also, f satisfies the Lipschitz condition:*

$$||f(t, \varphi_1) - f(t, \varphi_2)|| \le L||\varphi_1 - \varphi_2||$$

for all $\varphi_1, \varphi_2 \in C$ and every $t \ge 0$. If $u \in SAP_\omega(X)$ and $\frac{LM}{\mu} < 1$, then Eq. (1.1) has a unique S-asymptotically ω-periodic mild solution.

Proof. Define the operator $\Lambda : SAP_\omega(X) \to SAP_\omega(X)$ by

$$\Lambda u(t) = R(t)\varphi(0) + \int_0^t R(t-s) f(s, u_s) ds, \; t \ge 0.$$

Then

$$v(t) = \int_0^t R(t-s) f(s, u_s) ds$$

belongs to $SAP_\omega(X)$ by Lemma 2.13. For $x, y \in SAP_\omega(X)$,

$$\begin{aligned} ||\Lambda x(t) - \Lambda y(t)|| &\leq \int_0^t ||R(t-s)[f(s, x_s) - f(s, y_s)]|| ds \\ &\leq LM \int_0^t e^{-\mu(t-s)} ||x_s - y_s|| ds \\ &\leq \frac{LM}{\mu} ||x - y||_\infty. \end{aligned}$$

Therefore Λ is a contraction and there exists a unique fixed point $u \in SAP_\omega(X)$. This function u is an S-asymptotically ω-periodic mild solution of Eq. (1.1). □

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